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Cosmology: Exploring the Fate of our Universe

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***Abstract:** From 610 BCE hypotheses of Anaximander to the later conjectures of Plato, Aristotle and Ptolemy, the field of cosmology has been an important area of research for the scientific communitywho seek to improve understanding of the origin, evolution and ultimate fate of the Universe. This paper first introduces an outline of cosmology and reviews fundamental assumptions. It then explores the essential Physics, deriving and numerically solving the base case for the main Friedmann Equation. The last part of the paper includes plotting out the different fates of the Universe.*

1. WHAT IS COSMOLOGY?

The term cosmology is a branch of metaphysics and refers to the study of the nature of the Universe as a whole entity. In simple terms, cosmology can be thought of as the study of the cosmos as one entity, rather than separately analyzing the stars, black holes and galaxies that it is comprised of. Many of the earliest recorded scientific observations were about cosmology, and the pursuit of understanding has continued for over 5000 years. It began as a branch of theoretical physics through Einstein's 1917 paper 'Cosmological Considerations in the General Theory of Relativity', a mathematical framework that describes gravity as a consequence of the bending of space and time.

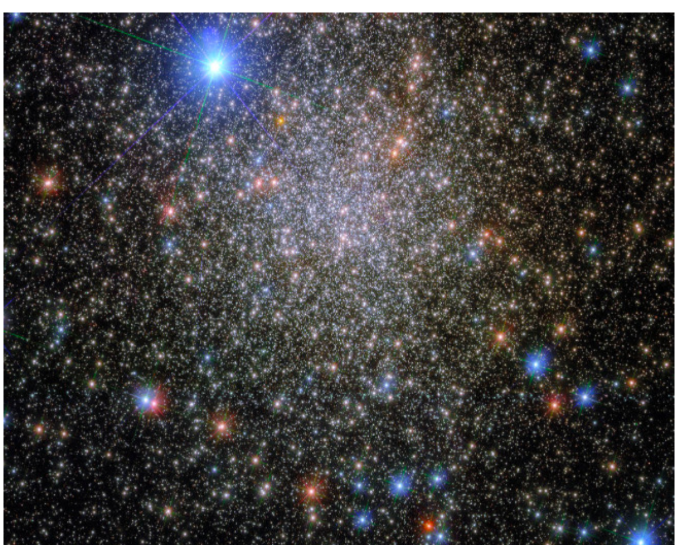

Fig. 1. Image of the cosmos (NASA)

It was developed in its early days particularly through the work of Lemaître, a pioneer in applying Einstein's theory of general relativity to cosmology. Cosmology has made enormous strides in the past century and has evolved from being the study of just the Universe to the search for an understanding about its overall architecture. This field asks big questions: Where did the Universe come from? Why does it have stars, galaxies and galaxy clusters? What's going to happen next?

Given the complexity of the Universe, this research review will only be considering the properties of an idealized, perfectly smooth, model Universe.

2. THE COPERNICAN PRINCIPLE: HOMOGENEITY AND ISOTROPY

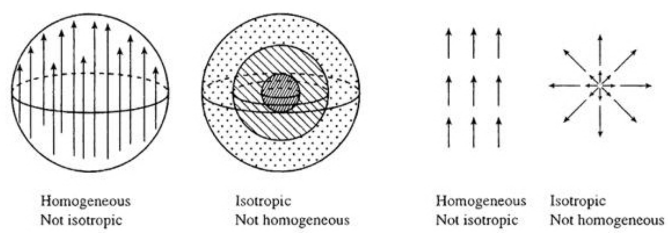

Fig. 2. Illustrations of how homogeneity and isotropy differ in three-dimensions and two-dimensions (University of Oregon).

The Copernican Principle states that the Earth is not the center of the Universe, and that, as observers, we don't occupy a special place. First stated by Copernicus in the 16th century, the idea is generally accepted by scientists and is an important assumption in many astronomical theories. This principle is based on the idea that the Universe, on large scales ($\geq$ 100 Mpc) is isotropic and homogeneous. Isotropy means that there are no special directions to the Universe (can be thought of as rotation-invariance) while homogeneous means that there are no special places in the Universe (can be thought of as translation-invariance). While these two definitions appear similar, they describe very different properties of the Universe. If the Universe is isotropic, this means that an observer would not see any difference in the structure of the Universe as they look in different directions. However, if the Universe is homogeneous, this means that, when viewed on the largest scales, the average density of matter is about the same in all places in the Universe and that the Universe is fairly smooth on large scales. In the study of cosmology, only the isotropy and homogeneity of the Universe on scales of millions of light-years (the distance it takes light to travel in one year is roughly 10^{18}cm in size) are considered.

3. MATHEMATICAL DESCRIPTION OF THE UNIVERSE

$$\boldsymbol{r}(\boldsymbol{t}) = \boldsymbol{a}(\boldsymbol{t})\,\boldsymbol{x}(\boldsymbol{t})$$

where r is the physical distance, a is a scale factor and x is the co-ordinate distance. The variables$r(t), a(t)$ and $x(t)$ are all functions of time, hence the (t).

$$H = \frac{\dot{a}}{a}$$

This equation represents Hubble constant, which in cosmology is the constant of proportionality in the relation between the velocities of remote galaxies and their distances. It expresses the rate at which the Universe is expanding.

For the derivation, we will be focusing on a large, zoomed in patch of the Universe with a bunch of massive particles that follow Hubble flow (i.e.,$\dot{x}$=0). The Hubble flow describes the motion of galaxies due solely to the expansion of the Universe.

$$v = \frac{dv}{dt} = \frac{d}{dt} = \dot{a}x + a\dot{x} = \dot{a}x$$

From this equation and the Hubble constant, we can derive Hubble's law:

$$v = \dot{a}x = \frac{\dot{a}}{a} \times ax = \frac{\dot{a}}{a} r = Hr$$

Therefore, the relationship $v = Hr$ is established.

The conservation of energy states that the total mechanical energy (kinetic energy plus potential energy) of a plant of mass m orbiting a sun of mass M can be given by the following equation:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

Using this equation and the previous ones above, we get:

$$E = \frac{1}{2}m(\dot{a}x) - \frac{GMm}{ax} = \frac{1}{2}m(Hax)^2 - \frac{Gm}{ax}\rho\frac{4}{3}\pi(ax)^3$$

where ρ is a constant representing energy density given that we are assuming homogeneity.

If we divide through by $\frac{1}{2}m(ax)^2$, we get:

$$H^2 - \frac{8\pi}{3}G\rho = \frac{2E}{m(ax)^2}$$

Since the left-hand side of the equation is independent of x, $\frac{2E}{m(ax)^2}$ must be independent of x as well.

From this, we can complete the last step to derive the 1^{st} Friedmann equation:

$$H^2 = \frac{8\pi}{3}G\rho - \frac{k}{a^2} + \lambda$$

where $k = \frac{2E}{mx^2}$ and λ is Einstein's cosmological constant. The cosmological constant is the energy density of space, or vacuum energy, that arises in Einstein's field equations of general relativity.

The 1^{st} Friedmann equation relates how the Universe expands, quantitively, to what makes up the matter and energy within it. We will explore this concept in due course.

What are the Friedmann Equations and Why are they of Importance?

The Friedmann equations are a set of equations in physical cosmology that govern the expansion of space in homogeneous and isotropic models of the Universe within the context of general relativity. They were first derived by Alexander Friedmann in 1922 from Einstein's field equations of gravitation for the Friedmann-Lemaître-Robertson-Walker metric and a perfect fluid (a fluid that can be completely characterised by its rest frame mass density ρ_m and isotropic pressure p). It is often thought of as being one of the most important equations in cosmology as it describes how, based on what is in the Universe, its expansion rate will change over time. It allows us to make predictions about the origin and fate of the Universe.

4. SOLVING THE FRIEDMANN EQUATIONS

The aim of solving the equation is to get an equation in terms of $a(t)$ by solving a differential equation. We will also be solving a continuity equation for ρ (energy density).

In order to proceed, we must recall the ideal gas law and the ideal gas energy equation:

$$PV = NkT$$

$$E = \frac{3}{2}NkT$$

where P is pressure, V is volume, T is temperature, n is the number of moles of gas and k is Boltzmann constant. The Boltzmann constant is the proportionality factor that relates the average relative kinetic energy of particles in a gas with the thermodynamic temperature of the gas.

$$dE = -PdV$$

$$d(\rho x^3 a^3) = -Pd(x^3 a^3)$$

$$d(\rho a^3) = -Pd(a^3)$$

$$\frac{d}{dt}(\rho a^3) = -P\frac{d}{dt}(a^3)$$

$$\dot{\rho}a^3 + p3a^2\dot{a} = -P3a^2\dot{a}$$

$$\dot{\rho} + 3\rho\frac{\dot{a}}{a} = -P3\frac{\dot{a}}{a}$$

From this, we can rearrange in order to get the continuity equation:

$$\dot{\rho} = -3H(P + \rho)$$

Given how complicated solving the Friedmann equation is, we need to make an important assumption: For a type of ρ (for e.g.,radiation, matter etc.), $P = w\rho$, where w is a dimensionless number equal to the ratio of the pressure of a perfect fluid to its energy density

Now, solving for $\rho(a)$:

$$\dot{\rho} = -3\frac{\dot{a}}{a}(P + \rho)$$

$$P = w\rho$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}(w\rho + \rho)$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}\rho(1 + w)$$

Let $\frac{3}{2}(1 + w) = \varepsilon$. Then, it follows that

$$\dot{\rho} = -2\varepsilon\rho\frac{\dot{a}}{a}$$

$$\frac{\dot{\rho}}{\rho} = -2\varepsilon\frac{\dot{a}}{a}$$

$$\frac{d}{dt}\ln(\rho(t)) = -2\varepsilon\frac{\dot{a}}{a}$$

$$\frac{d}{dt}\ln(\rho) = -2\varepsilon\frac{d}{dt}\ln(a)$$

$$\int_{t_0}^{t}\frac{d}{dt}\ln(\rho)\,dt = -2\varepsilon\,(\ln(a(t)) - \ln(a(t_0)))$$

$$\ln\left(\frac{\rho(t)}{\rho_0}\right) = -2\varepsilon\ln(\frac{a(t)}{a_0})$$

$$\frac{\rho(t)}{\rho_0} = (\frac{a(t)}{a_0})^{-2\varepsilon}$$

From this, we can deduce the relationship $\rho \propto a^{-2\varepsilon}$

5. PRIMARY FORMS OF ENERGY DENSITY

The simplest example of a component that is of this form is a set of massive particles with negligible relative velocities, known as 'dust' or 'matter'. The energy density of such particles is given by their number density multiplied by their rest mass. As the Universe expands, the number density is inversely proportional to the volume, while the rest masses are constant, thus yielding $\rho_m \propto a^{-3}$. For relativistic particles, known as 'radiation' (in cosmology, this term is used to signify any relativistic species, not only photons or massless particles), the energy density is the number density times the particle energy, and the latter is proportional to a^{-1} (redshifting as the Universe expands); the radiation energy density therefore scales as $\rho_R \propto a^{-4}$. Vacuum energy does not change as the Universe expands, so $\rho_\lambda \propto a^0$.

The most popular equations of state for cosmological energy sources can therefore be summarized as follows:

ρ_i	w_i	ε_i	$\rho_i \propto a^{-2\varepsilon_i}$
Matter (dust)	0	$\frac{3}{2}$	$\rho_m \propto a^{-3}$
Radiation	$\frac{1}{3}$	2	$\rho_R \propto a^{-4}$
'curvature'- k	$-\frac{1}{3}$	1	$\rho_k \propto a^{-2}$
Vacuum (i.e., dark energy)	-1	0	$\rho_\lambda \propto a^0$

6. COSMOLOGICAL INITIAL CONDITIONS VALUES

The densities below are in natural units and are the Planck densities.

$$\rho_m = 5.0e^{-124}$$

$$\rho_R = 0$$

$$k = 0$$

$$cc\ (cosmological\ constant) = 9.9e^{-123}$$

7. QUICK SOLUTION TO FRIEDMANN EQUATIONS

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \lambda$$

$$H^2 = \frac{8\pi G}{3}\left(\frac{\rho_{M0}}{a^3} + \frac{\rho_{R0}}{a^4}\right) - \frac{k}{a^2} + \lambda$$

Since $H = \frac{\dot{a}}{a}$, $\dot{a} = Ha$

$$\therefore \dot{a} = a\,(\frac{8\pi G}{3}\left(\frac{\rho_{M0}}{a^3} + \frac{\rho_{R0}}{a^4}\right) - \frac{k}{a^2} + \lambda)^{\frac{1}{2}}$$

Natural Units and Early Universe Conditions

An approximation that we make is that things are very dense and hot. Classical descriptions break down at the Planck scale. The Planck scale sets the Universe's minimum limit, beyond which the laws of physics break. This scale is defined by taking the constants of nature and combining them in such a way that their units combine to give a length.

Below is a derivation of the Plank scale:

$$L_p = (\frac{hG}{2\pi c^3})^{\frac{1}{2}} = 1.6 \times 10^{-35}\, m$$

$$m_p = (\frac{hc}{2\pi G})^{\frac{1}{2}} = 2.2 \times 10^{-8}\, kg$$

$$t_p = (\frac{hG}{2\pi c^5})^{\frac{1}{2}} = 5.4 \times 10^{-44} s$$

$$T_p = (\frac{hc^5}{G{k_B}^2})^{\frac{1}{2}} = 1.4 \times 10^{32}\, K$$

Simulation Proposal- Initial Conditions

$$\rho_{m0} = \rho_{R0} = \frac{m_p}{{l_p}^3} = \rho_p$$

Using natural units, $\rho_{m0} = \rho_{R0} = 1$

Different Possible Fates of the Universe (Based on the curvature of the Universe)

The Friedmann equation which models the expanding Universe has a parameter k called the curvature parameter which is indicative of the rate of expansion and whether or not that expansion rate is increasing or decreasing. It indicates the future of the Universe. Based on different values of k, there are thought to be three major possible fates of the Universe.

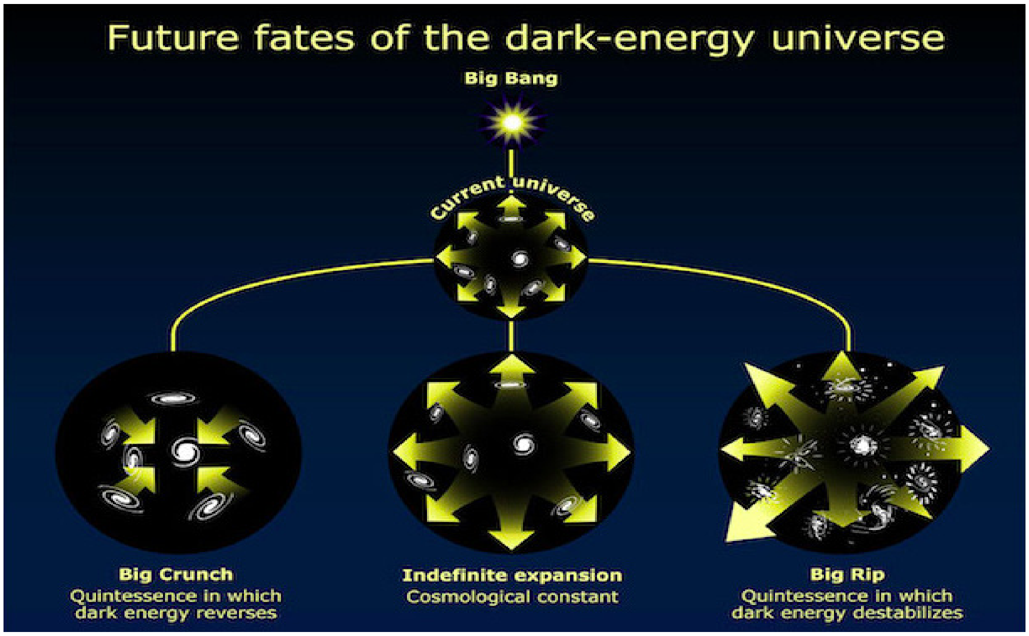

Fig. 3. Three possible fates of the dark-energy Universe (The European Space Agency)

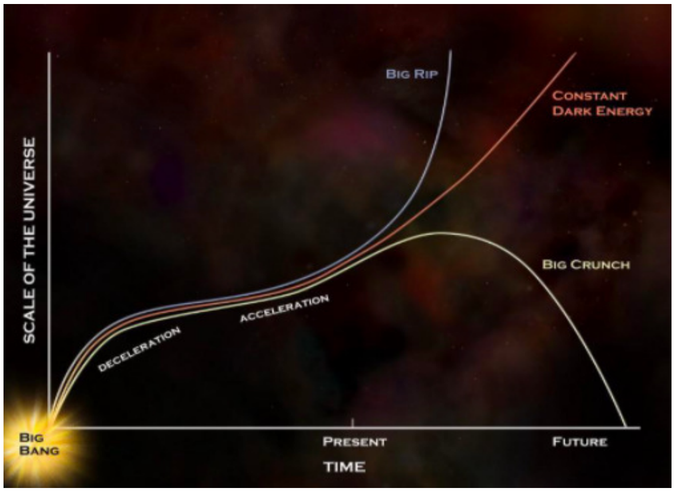

Fig. 4. A graphical model of the different ways dark energy could evolve into the future. The three trend lines have been generated through the use of a non-zero cosmological constant (NASA).

If $k = 0$, then the density is equal to a critical value at which the Universe will expand forever at a decreasing rate. This is often referred to as the Einstein-de Sitter Universe in recognition of their work in modelling it. This $k = 0$ condition can be used to express the critical density in terms of the present value of the Hubble parameter. Indefinite expansion: in a flat Universe with no dark energy, the Universe will expand forever but at a continually decelerating rate. If dark energy is around, then at first the expansion of the Universe initially slows down (due to gravity) but eventually speeds up. The ultimate fate of the Universe is the same as if it were open. The Universe's density and critical density are equal.

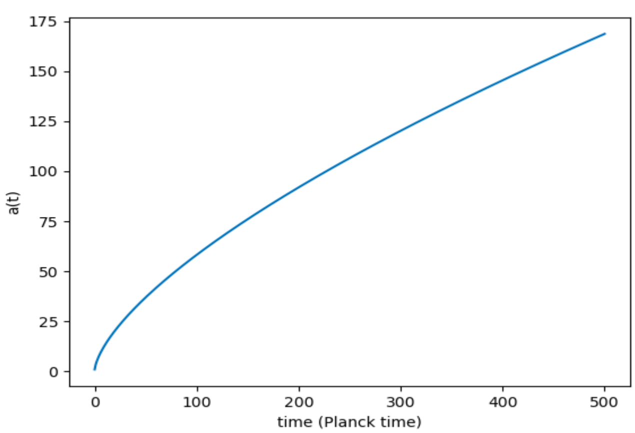

Fig. 5. A plot of a(t)against Planck time when $k = 0$

For $k > 0$, the density is high enough that the gravitational attraction will eventually stop the expansion and it will collapse backward to a 'Big Crunch'. This kind of Universe is described as being a closed Universe, or a gravitationally bound Universe. In a closed Universe, where the density of the Universe is more than five atoms of hydrogen per cubic metre (a critical density), there is no repulsive effect of dark energy and gravity eventually halts the expansion of the Universe. Beginning to then contract, all of the matter in the Universe collapses to a point.

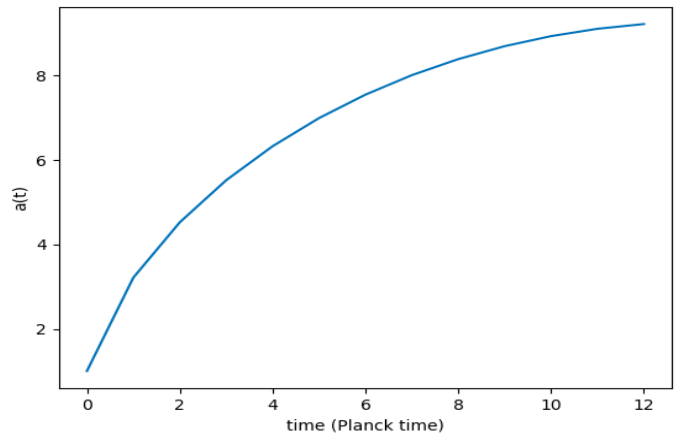

Fig. 6. A plot of a(t) against Planck time when $k = 1$

For $k < 0$, the Universe expands forever, there not being sufficient density for gravitational attraction to stop the expansion. If the geometry of space is open (and curved like a horse saddle), the Universe will continue to expand forever, whether there is dark energy present or not. If it is, then dark energy will drive the expansion. The result is the Big Freeze or the Big Rip.

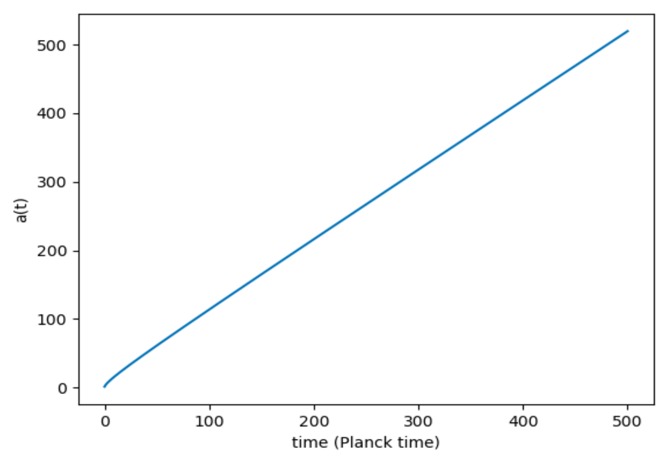

Fig. 7. A plot of a(t) against Planck time when $k = -1$

8. CODE FOR FATES OF THE UNIVERSE PLOTS

Pasted below are screenshots of the Jupyter code used to generate the plots in Fig. 5., Fig. 6.and Fig. 7.The code below is an example for when $k = 0$.

In [15]:
```
%matplotlib notebook
import numpy as np
from scipy.integrate import odeint
import matplotlib.pyplot as plt
#import seaborn as sns
#sns.set_style("whitegrid")
```

Setting up the mathematics

In [17]:
```
# function that returns dy/dt
def derivative(a,t):
    """
    Derivative function
    Input: (a value, t value)
    Output: Derivative value da/dt from (a,t)
    """
    rho_m0 = 1.0
    rho_r0 = 1.0
    k = 0
    cc = 0

    dadt = a * np.sqrt( (8*np.pi/3)*(rho_m0/ a**3 + rho_r0/a**4) - k/a**2 + cc )
    return dadt

# initial condition
a0 = 1.0

# time points
t = np.linspace(0,500,501)
```

Solving and plotting the solution

In [4]:
```
# solve ODE
a = odeint(derivative,a0,t)

# plot results
plt.plot(t,a)
plt.xlabel('time (Planck time)')
plt.ylabel('a(t)')
#plt.show()
```

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